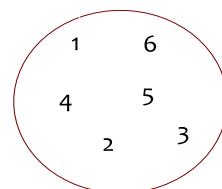


## Binomial Coefficients

These are sets of integers which are important in many areas of mathematics. To start with we will ignore the name, and see how they arise. We come back to these numbers in the 'binomial expansion'.

### Choosing things from sets

Suppose we have a set with six elements, which we label here 1 to 6 just so we can say which is which:



Suppose we want to pick out one of these. How many ways can we do this? Obviously 6, and the choices are 1,2,3,4,5 or 6.

Suppose we want to pick out pairs of items. One choice is {1,2} Another is {4,6}. How many are there altogether? Well we can pick the first in 6 ways, and having made the first choice, we have a choice of 5 for the second. So there are 30 pairs. Here they are:

1,2    1,3    1,4    1,5    1,6  
2,1    2,3    2,4    2,5    2,6  
3,1    3,2    3,4    3,5    3,6  
4,1    4,2    4,3    4,5    4,6  
5,1    5,2    5,3    5,4    5,6  
6,1    6,2    6,3    6,4    6,5

Notice that there are pairs like {1,3} and {3,1} which have the same elements but in different order. Do they count as being different choices? We can do it both ways. **If the order matters, they are permutations.** So {1,3} and {3,1} are different permutations.

**If the order does not matter, they are combinations.** So {1,3} and {3,1} are the same combination.

In this example, our 30 pairs are permutations. For each pair {a,b} there is another pair {b,a}. So the number of combinations, where the order does not matter, is 15. We got this by

$$\frac{6 \times 5}{2}$$

Suppose we choose 3 from 6. How many ways? We can choose the first in 6 ways, the second from any of the remaining 5, and the third in 4 ways. So we have 6X5X4 permutations. How many combinations? For each set of 3 (such as {a,b,c}) how many ways would these be ordered? We have 3 choices of the first (a or b or c), and 2 choices for the second, and just one option for the third. In fact these are {a,b,c}, {a,c,b}, {b,a,c}, {b,c,a}, {c,a,b} and {c,b,a}. So we have 3X2 orderings of each permutation. So to choose 3 from 6, the number of combinations is

$$\frac{6 \times 5 \times 4}{3 \times 2 \times 1}$$

What about choosing 5 from 6? We have 6X5X4X3X2 permutations, and each of these is ordered in 5X4X3X2X1 ways. So the number of choices is

$$\frac{6 \times 5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2 \times 1}$$

which equals 6. This checks out, since choosing 5 is the same as choosing 1 (to leave out), and there are 6 choices of 1.

## Choosing k from n

To generalise this, we are choosing k items from a set of n. We can choose the first in n ways, the second in n-1, the third in n-2, and the k<sup>th</sup> in n-k+1. So the number of permutations is

$$n.(n-1).(n-2)..(n-k+1)$$

But for each of permutation, it can be ordered in several ways – we can have k items as the first, k-1 as the second, k-3 as the third and so on. So the number of orderings is k(k-1)(k-2)..1. So the number of combinations is

$$\frac{n.(n-1).(n-2)..(n-k+1)}{k.(k-1).(k-2)..1}$$

We can write this more easily using factorials. The factorial of n is the product n.(n-1).(n-2).(n-3)..1. For example, factorial 3 = 3X2X1 = 6. Factorial 4 is 4X3X2X1 = 24. Factorial n is written as n!

So we can write our formula as

$$\frac{n.(n-1).(n-2)..(n-k+1)(n-k).(n-k-1)..1}{(n-k).(n-k-1).(n-k-2)..1.k.(k-1).(k-2)..1} = \frac{n!}{(n-k)!k!}$$

This is usually written as  $\binom{n}{k}$ , or  ${}_nC_k$  read as 'n choose k'. So

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Two special values are

$$\binom{n}{0} = 1$$

and

$$\binom{n}{n} = 1$$

## Some values

If we work out a few values we get this.

k	0	1	2	3	4	5
n						
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

If we arrange these as a triangle:

```

      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
 1 5 10 10 5 1
```

then each number is the sum of the two above it. This is called **Pascal's triangle**.