

Binomial expansions

Here are some examples of binomials:

$(x+y)^2$	$(x+y)^3$	$(x+y)^4$	$(x+y)^{15}$
$(1+x)^2$	$(2x+1)^3$	$(1-x/2)^4$	$(y+x)^7$

So a **binomial** is $(A+B)^n$ where A and B are expressions, and n is a positive integer (in fact n can be negative or non-integral, but here we will just deal with positive integers).

Binomials are so-called because they are 'two names', added and raised to a power.

A **binomial expansion** is what you get when a binomial is multiplied out – the expansion of a binomial.

Some examples

$$1. (x+y)^2 = x^2+xy+xy+y^2 = x^2+2xy+y^2$$

$$2. (x-1)^2 = x^2 - 2x + 1$$

$$3. (1+2y)^2 = 1 + 4y + 4y^2$$

$$4. (x+y)^3 = (x+y)(x+y)^2 = (x+y)(x^2+2xy+y^2) = x^3+2x^2y+xy^2 + yx^2+2xy^2+y^3 \\ = x^3+3x^2y+3xy^2+y^3$$

This one is slightly long and likely to go wrong.

$$5. (x+y)^6 =$$

$$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

I could do this one because I know the trick.. If you find the following a bit hard, [look here](#).

The binomial expansion

The trick is to work this out without actually multiplying out all the brackets.

Suppose we look at $(x+y)^3$. This is

$$(x+y)(x+y)(x+y) = x^3+3x^2y+3xy^2+y^3$$

One term in this is x^3 . We get this by when we multiply the factors, we chose x for each one. There is only one way to do this, so the term is just $1x^3$.

Another term is y^3 . This comes from choosing y from each factor, and again there is only one way to do this.

What about x^2y ? This comes from choosing x from 2 factors, and y from 1. But we can do this in more than 1 way – in fact 3 ways. So we get $3x^2y$.

Similarly for xy^2 – choosing x from 1 and y from 2 factors. We can do this in 3 ways, so the term is $3xy^2$.

Expanding $(x+y)^n$

This is

$$(x+y)(x+y)(x+y)\dots(x+y) \quad (\text{n factors})$$

The first term in this will be x^n

The second term will include $x^{n-1}y$ (times by something)

The third term will include $x^{n-2}y^2$

The fourth term will include $x^{n-3}y^3$

The m^{th} term will include $x^{n-m+1}y^{m-1}$

The last term will be y^n

The powers of x go down, as the powers of y go up. For example in $(x+y)^5$, the terms are

$$x^5, x^4y, x^3y^2, x^2y^3, xy^4 \text{ and } y^5$$

But each term is also multiplied by a number. How do we get that?

Binomial coefficients

In the expansion of $(x+y)^5$, the first term includes x^5 , and there is only one way to get that – choosing x , not y , from each factor. So the actual term is x^5

The second term is x^4y . We can get this in several ways – we must choose x from 4 factors (and y from the other one). How many ways are there to choose 4 items from 5? The answer is 5. So this term is $5x^4y$.

But we have seen this question before. This is 'n choose k', choosing 4 items from 5, and is written

$$\binom{5}{4}. \text{ Our formula for this is } \frac{n!}{(n-k)!k!} = \frac{5!}{(1)!4!} = 5$$

The next term includes x^3y^2 . The factor multiplying this is the number of ways of choosing 3 from 5, which is $\binom{5}{3} = \frac{n!}{(n-k)!k!} = \frac{5!}{(2)!3!} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$. So the next term is $10x^3y^2$.

And so on. The whole thing is

$$\binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

This is why **binomial coefficients** are so called. They are the coefficients multiplying the terms in the expansion of a binomial expression.

Slightly easier

Often we only need to work out the first few terms in an expansion. We just need to get two things correct –

- The powers of the x and y
- The coefficient multiplying the term

The first term is just x to the power of the expansion. The next term is 1 less for x, and y for y. The next is 2 less for x, and 2 for y.

For example, the expansion of $(x+y)^{20}$ goes

$$x^{20} \quad x^{19}y \quad x^{18}y^2 \quad x^{17}y^3$$

and so on.

The first term has coefficient 1. The second has coefficient 20. The next is $20 \times 19 / (1 \times 2)$. Like this

$$x^{20} + 20x^{19}y + \frac{20 \cdot 19}{1 \cdot 2}x^{18}y^2 + \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3}x^{17}y^3 + \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4}x^{16}y^4 + \dots$$

We have called the two terms in the binomial x and y. In fact they might be anything, and we need to replace them correspondingly. For example, if we want $(1+x)^{20}$ we have to replace x by 1, and y by x. So the expansion of $(1+x)^{20}$ starts:

$$1^{20} + 20 \cdot 1^{19}x + \frac{20 \cdot 19}{1 \cdot 2}1^{18}x^2 + \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3}1^{17}x^3 + \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4}1^{16}x^4 + \dots$$

$$1 + 20x + \frac{20 \cdot 19}{1 \cdot 2}x^2 + \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3}x^3 + \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots$$

Worked example

(Edexcel C2 May 2006) Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2+x)^6$, giving each term in its simplest form.

Answer: (We must write 2 in place of x, and x in place of y)

$$\text{Result} = 2^6 + 6 \cdot 2^5x + \frac{6 \cdot 5}{1 \cdot 2}2^4x^2$$

$$= 64 + 6 \cdot 32x + 15 \cdot 16x^2$$

$$= 64 + 192x + 240x^2 + \dots$$

Using Pascal's Triangle

It is sometimes quicker to use Pascal's Triangle, which is

				1								
				1		1						
			1		2		1					
		1		3		3		1				
		1	4		6		4		1			
	1		5		10		10		5			
1		6		15		20		15		6		1

These are just the binomial coefficients. If you forget the numbers, work them out again – each number is the sum of the 2 above it.

Each row is the coefficients in the expansion. For example the row highlighted applies to a binomial to the fourth power – in other words

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

This is a quick way for low powers.

Worked example

(AQA C2 Jan 2009)

(a) By using the binomial expansion, or otherwise, express $(1+2x)^4$ in the form

$$1 + ax + bx^2 + cx^3 + 16x^4$$

where a , b and c are integers. (4 marks)

(b) Hence show that $(1 + 2x)^4 + (1 - 2x)^4 = 2 + 48x^2 + 32x^4$. (3 marks)

(c) Hence show that the curve with equation

$$y = (1 + 2x)^4 + (1 - 2x)^4$$

has just one stationary point and state its coordinates.

Answer: (The 'or otherwise' means we could just multiply this out, but the binomial expansion is quicker and less prone to mistakes. Since this is just the 4th power, Pascal's triangle is good. We must think of x as 1, and y as $2x$)

$$\text{a) } (1+2x)^4 = 1^4 + 4 \cdot 1^3(2x) + 6 \cdot 1^2(2x)^2 + 4 \cdot 1^1(2x)^3 + (2x)^4$$

$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

b) Similarly

$$(1-2x)^4 = 1^4 + 4 \cdot 1^3(-2x) + 6 \cdot 1^2(-2x)^2 + 4 \cdot 1^1(-2x)^3 + (-2x)^4$$

$$= 1-8x+24x^2-32x^3+16x^4$$

$$\text{So } (1+2x)^4 + (1-2x)^4 =$$

$$1+8x+24x^2+32x^3+16x^4 + 1-8x+24x^2-32x^3+16x^4$$

$$= 2+48x^2+32x^4$$

c) So

$$dy/dx = 96x+128x^3 = x(96+128x^2)$$

At a stationary point, $dy/dx = 0$

$$\text{So } x(96+128x^2)=0$$

so $x=0$ or $96+128x^2 = 0$, for which there is no real solution.

At $x=0$, $y = 2$. So the stationary value is at (0,2)