Exponentials and logarithmic functions

Exponential functions

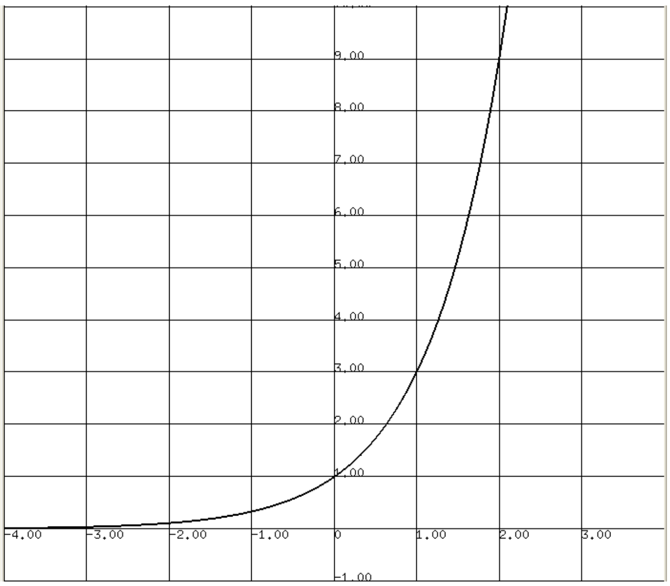
These are 2^x or 3^x or 4^x – in general, a^x where a is a constant. Remember it is *not* for example x^3, but ‘the other way round’, like 3^x.

Here are some values for 3^x

x	-2	-1	0	1	2	3
3^x	1/9	1/3	1	3	9	27

Why is $3^0 = 1$? Because we have the Law of Exponents, like $3^1 \times 3^2 = 3^{1+2} = 3^3$. It would be good if this also worked for a power of zero, like $3^1 \times 3^0 = 3^{1+0} = 3^1$. This means that 3^0 should be defined to be 1. This works for (almost) anything – anything to the power of zero is one. The weird item is 0^0, about which there is some debate.

The graph of 3^x is like this:



All graphs of a^x are like this, having all values positive, tending to 0 at $-\infty$, going though (0,1), and always increasing.

Logarithms

A logarithm function (often abbrieviated to ‘log’) is the inverse of an exponential function. In other words, if

$y = a^x$, then $x = \log_a y$

The constant 'a' is the base of the log. In other words there is not just one log function, but lots of them, to different bases.

For example, $81 = 9^2$, so $2 = \log_9 81$

Or $8=2^3$, so $3=\log_2 8$

Or $100=10^2$, so $2=\log_{10} 100$

What is $\log_8 64$? This is $\log_8 8^2$, so the answer is 2

The log of a number is what power you have to raise the base to to get the number.

For example, what is the log to base 10 of 1000? Answer is 3, because $10^3=1000$.

Laws of logs

$\log_a a=1$ because $a^1=a$

$\log_a 1=0$ because $a^0=1$

$\log a + \log b = \log ab$

The reason for this is as follows. Suppose we are working in base c. Call $\log_c a=n$ and $\log_c b=m$.

Then $a=c^n$ and $b=c^m$. So $ab=c^n c^m=c^{n+m}$

So $\log_c ab = \log_c c^{n+m} = n+m = \log_c a + \log_c b$

$n \log a = \log a^n$

because $\log a^n = \log a.a.a..a$ (n factors) $= \log a + \log a ..+\log a$ (n terms) $= n \log a$

$\log 1/a = - \log a$

because $\log 1/a = \log 1.a^{-1}=\log 1 – \log a = 0 – \log a = -\log a$

$\log a – \log b = \log a/b$

because $\log a/b = \log a b^{-1} = \log a + \log b^{-1} = \log a – \log b$

Worked Examples

1. Solve $4^x=100$ to 4 s.f.

The 'to 4 s.f.' suggests we will need a calculator.

Take logs to base 10 of both sides (could be any base – but our calculator can do logs to base 10, and 100 is a power of 10:

$\log_{10} 4^x = \log_{10} 100$

$x \log_{10} 4 = 2$

$x = 2/\log_{10} 4 = 2/ 0.6021 = 3.322$ to 4 sig figs

2. Write each of the following as a single log:

a) $\log_a x+3\log_a y -1/2 \log_a x$

b) $\log_{10} x-1$

a) $\log_a x+3\log_a y -1/2 \log_a x =$

$\log x + \log y^3 – \log \sqrt{x} = \log x + \log y^3 + \log 1/\sqrt{x}$

$\log (x.y^3. 1/\sqrt{x})$

$=\log (y^3\sqrt{x})$

b) $\log_{10} x-1 =$

$\log_{10} x-\log_{10} 10 =$

$\log_{10}(x/10)$

3) Evaluate:

a) $\log_2 8$ Answer = $\log_2 2^3 =3$

b) $\log_9 3$ Answer = $\log_9 9^{1/2} = 1/2$

c) $2\log_4 2$ Answer = $2\log_4\sqrt{4} =2.1/2 =1$

4) (Edexcel C2 May 2006) (i) Write down the value of $\log_6 36$.

Answer: 2 (because it is $\log_6 6^2$)

(ii) Express $2 \log_a 3 + \log_a 11$ as a single logarithm to base a.

Answer: $2 \log_a 3 + \log_a 11 = \log_a 9 + \log_a 11 = \log_a 99$

5. (Edexcel C2 June 2005) Solve

(a) $5^x = 8$, giving your answer to 3 significant figures,

Answer: $x \log_{10} 5=\log_{10} 8$

so $x = \log_{10} 8/\log_{10} 5 = 1.29$ to 3 sig figs (by calculator)

(b) $\log_2 (x + 1) – \log_2 x = \log_2 7$.

Answer: $\log_2 ((x+1)/x) = \log_2 7$

so $1+1/x=7$

$x+1=7x$

$x=1/6$

6 (OCR C2 June 2009) Use logarithms to solve the equation $7^x = 2^{x+1}$, giving the value of x correct to 3 significant figures.

Answer: Take logs to base 10 of both sides:

$\log 7^x = \log 2^{x+1}$

so $x \log 7 = (x+1) \log 2$

$x \log 7 = x \log 2 + \log 2$

$x (\log 7 – \log 2) = \log 2$

$x= \log 2 / (\log 7 – \log 2)$

$= 0.3010 / (0.8451 – 0.3010)$

= 0.533 to 3 sig figs

7) (AQA C2 Jan 2009((a) Write each of the following in the form $\log_a k$, where k is an integer:

(i) $\log_a 4 + \log_a 10$; (1 mark)

Answer: $\log_a 40$

(ii) $\log_a 16 - \log_a 2$; (1 mark)

Answer: $\log_a 8$

(iii) $3 \log_a 5$. (1 mark)

Answer: $\log_a 125$ (5 to the power of 3

(b) Use logarithms to solve the equation $(1.5)^{3x}=7.5$, giving your value of x to three

decimal places. (3 marks)

Answer: take logs to base 10 both sides:

$3x \log 1.5 = \log 7.5$

$x = \log 7.5 / (3 \log 1.5)$

=1.656 to 3dps

(c) Given that $\log_2 p = m$ and $\log_8 q = n$, express pq in the form 2^y , where y is an

expression in m and n. (3 marks)

Answer:

$p=2^m$

$q=8^n = (2^3)^n = 2^{3n}$

so $pq = 2^m 2^{3n} = 2^{3n+m}$