

Sequences and series

Sequences

A **sequence is a list of numbers** (actually, they don't have to be numbers).

Here is a sequence: 1, 4, 9, 16

The order makes a difference, so 16, 9, 4, 1 is a different sequence.

The items in a sequence are often referred to using subscripts, like u_1 , u_2 , u_3 and so on. The items are called **terms**.

We often define a sequence by giving an algebraic expression for the n^{th} term.

So for 1, 4, 9, 16, the sequence is

$$u_n = n^2$$

Some sequences have a fixed length and have a last term, while others 'go on forever'.

A **series** is the terms in a sequence *added up*.

Exercise

1. Write down the first three terms in the sequence $u_n = n^3$
2. Write down the first three terms in the sequence $u_n = 1/n$
3. Write down the first three terms in the sequence $u_n = 2n + 1$

Arithmetic Progressions

Look at the sequence 7, 9, 11, 13, 15, 17..

What is the next term?

This sequence starts at 7, and each term is 2 more than the previous one. This is an example of an arithmetic progression.

In an arithmetic progression, each term is formed by adding a fixed amount to the previous term.

The fixed amount is called the **common difference**.

Arithmetic progression is usually abbreviated to '**ap**'

So the terms in an ap are

$a, a+d, a+2d, a+3d, a+4d..$

where a is the first term, and d is the common difference

$$\text{So } u_n = a + (n-1)d$$

An **arithmetic series is the sum of the terms in an ap.** For example,

$$10 + 12 + 14 + 16 + 18 + 20$$

is an arithmetic series with six terms.

Sum of an ap

If we add up the terms in an ap, can we find a formula to work this out?

Call the sum of the first n terms S_n . Then

$$S_n = a + a+d + a+2d + a+3d \dots a+(n-1)d$$

write this down again in reverse

$$S_n = a+(n-1)d + a+(n-2)d + a+(n-3)d \dots a$$

add these term by term

$$2S_n = 2a+(n-1)d + 2a+(n-1)d + 2a+(n-1)d \dots + 2a+(n-1)d$$

(In the first, terms go up by d , and in the second, terms go down by d . So each pair of terms is the same, and we have n of them). So

$$2S_n = n(2a + (n-1)d)$$

$$S_n = n/2 (2a+(n-1)d) = n/2 (a + a+(n-1)d)$$

$$= n/2(\text{first term} + \text{last term})$$

Worked example

1. (AQA C2 Jan 2009) The 25th term of an arithmetic series is 38.

The sum of the first 40 terms of the series is 1250.

(a) Show that the common difference of this series is 1.5.

(b) Find the number of terms in the series which are less than 100.

Answer: (We have two unknowns, the first term and the common difference. The question gives us two pieces of information. So we expect to get two simultaneous equations with two unknowns which we can solve)

$$38 = a + 24d \quad (1).. (25^{\text{th}} \text{ term})$$

$$1250 = 20(a + a + 39d) \quad (2).. \text{ (sum of 40 terms)}$$

$$\text{From (1), } a = 38 - 24d$$

Substitute this in (2)

$$1250 = 20(76 - 48d + 39d)$$

$$125/2 = 76 - 9d$$

$$9d = 76 - 62\frac{1}{2} = 13\frac{1}{2}$$

$$d = 1\frac{1}{2}$$

Substitute this in (1)

$$38 = a + 36$$

$$a = 2$$

b) The n th term is $a(n-1)d$. Find n such that this is 100:

$$100 = 2 + (n-1)3/2$$

$$98 = (n-1)3/2$$

$$n-1 = 196/3 = 65.33$$

$$n = 66.33$$

$$\text{The 66}^{\text{th}} \text{ term is } 2 + 65 \times 3/2 = 99\frac{1}{2}$$

$$\text{The 67}^{\text{th}} \text{ term is } 2 + 66 \times 3/2 = 101$$

So 66 terms are less than 100

Geometric progressions

Look at this sequence

3, 6, 12, 24, 48

Each term is the previous one *multiplied* by a constant factor ($= 2$). This differs from an ap, where terms were formed by *adding* to the previous term. Such a sequence is called a geometric progression, or gp, and the constant multiplying factor is called the **common ratio**.

The terms in a gp are:

$$a, ar, ar^2, ar^3..$$

where a is the first term and r is the common ratio. The n th term is ar^{n-1} .

Sum of a gp

What do we get if we add up the first n terms of a gp?

$$S_n = a + ar + ar^2 \dots ar^{n-1}$$

multiply by r

$$rS_n = ar + ar^2 \dots + ar^{n-1} + ar^n$$

subtract

$$S_n(1-r) = a - ar^n$$

so

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

The common ratio may be positive or negative. In the latter case terms will alternate in sign – eg if $a=2$ and $r=-3$ the sequence is

2, -6, 18, -48..

If r is greater than 1, the terms will increase in size. If $|r|$ is less than 1, the terms get smaller. For example, if $a=1$ and $r=1/2$, the sequence is

1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$..

If $|r| < 1$, we can sum an infinite number of terms in the series. If $|r| < 1$, $u_n \rightarrow 0$ as $n \rightarrow \infty$, and so the sum to infinity of a gp is

$$S = \frac{a}{(1-r)}$$

Worked example

(Edexcel C2 May 2006) A geometric series has first term a and common ratio r . The second term of the series is 4 and the sum to infinity of the series is 25.

(a) Show that $25r^2 - 25r + 4 = 0$.

(b) Find the two possible values of r .

(c) Find the corresponding two possible values of a .

(d) Show that the sum, S_n , of the first n terms of the series is given by

$$S_n = 25(1 - r^n).$$

Given that r takes the larger of its two possible values,

(e) find the smallest value of n for which S_n exceeds 24.

Answer: (Two unknowns (a and r) and two pieces of information, leading to two simultaneous equations)

$$a) 4 = ar \quad (1) \text{ (second term)}$$

$$25 = a/(1-r) \quad (2) \text{ (sum to infinity)}$$

From (2)

$$a = 25(1-r)$$

Substitute this in (1)

$$4 = 25(1-r)r$$

$$\text{so } 4 = 25r - 25r^2$$

$$25r^2 - 25r + 4 = 0$$

$$b) (5r - 1)(5r - 4) = 0$$

$$\text{so } r = 1/5 \text{ or } r = 4/5$$

c) Using (1),

$$\text{if } r = 1/5, a = 4/r = 20$$

$$\text{if } r = 4/5, a = 5$$

$$d) S_n = a + ar + ar^2 \dots ar^{n-1}$$

multiply by r

$$rS_n = ar + ar^2 \dots + ar^{n-1} + ar^n$$

subtract

$$S_n(1-r) = a - ar^n$$

so

$$S_n = a(1-r^n)/(1-r)$$

$$\text{if } r = 1/5 \text{ and } a = 20$$

$$S_n = 20(1-r^n)/4/5$$

$$= 25(1-r^n)$$

$$\text{if } r = 4/5 \text{ and } a = 5$$

$$S_n = 5(1-r^n)/1/5$$

$$= 25(1-r^n)$$

$$d) r=4/5$$

$$24 = 25(1-(4/5)^n)$$

$$24/25 = (1-(4/5)^n)$$

$$(4/5)^n = 1/25$$

$$n \log 0.8 = \log 0.04$$

$$n = -1.3979 / -0.096910 = 14.425$$

(So n is 15 . Check actual values..)

$$\text{If } n=14,$$

$$S_n = 25(1-r^n) = 25(1-0.043980) = 23.9005$$

$$\text{If } n=15,$$

$$S_n = 25(1-r^n) = 25(1-0.035184) = 24.12039$$

so n=15