

The Chain Rule – Function of a function

How do we differentiate something like $\sin(x^2)$? We know the derivative of $\sin(x)$, but what about $\sin(x^2)$?

In fact this is what is called a 'function of a function'.

Suppose we wanted to evaluate this for say $x=2$. We would need to first work out x^2 to get 4, then work out $\sin(4)$.

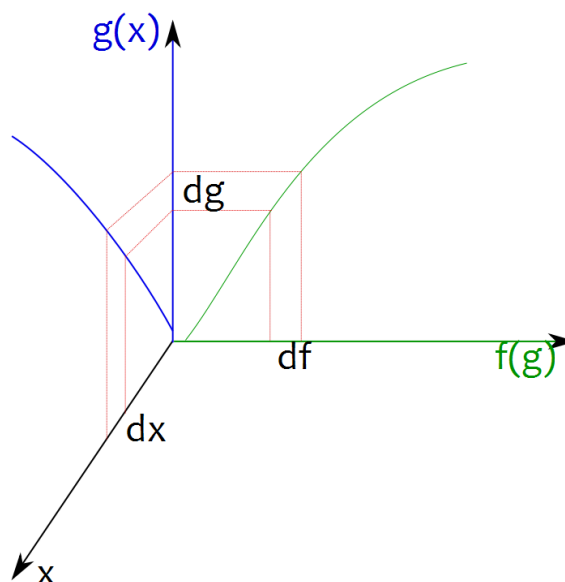
So reading from left to right, we first have sine, then x^2 . But to work it out, we *first* do the *inside* function x^2 , *then* do the sine.

If we named $g(x)=x^2$ and $f(g)=\sin(g)$, we are actually trying to differentiate $f(g(x))$. This is a function (f) of a function (g).

A visual approach

We can visualise this as two graphs in 3D. We have (in blue) the function $g(x)$, drawn vertically. The function f is drawn sideways in green. The function f is a function of g , and g is a function of x .

When x changes by an amount dx , g changes by dg and f changes by df .



By definition of the derivative,

$$dg = g' dx$$

$$d(f(g)) = f' dg = f' g' dx$$

So

$$\frac{d(f(g))}{dx} = f' g'$$

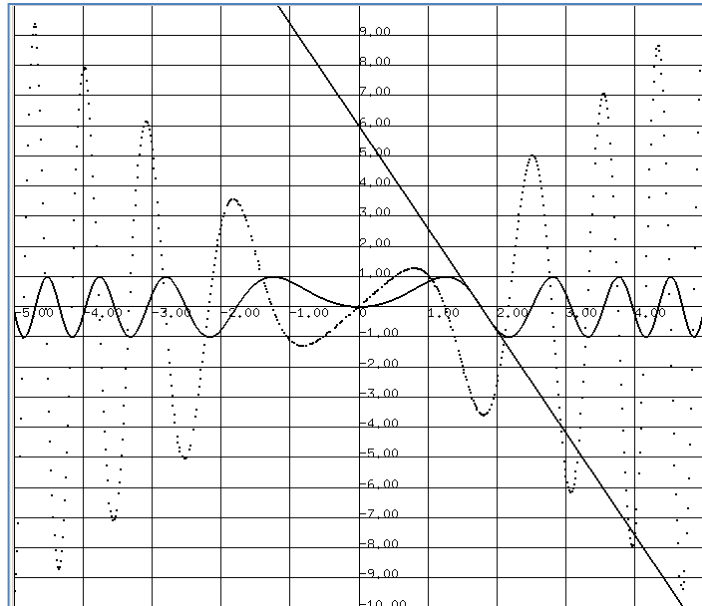
This is called the **chain rule**.

In our example, $g=x^2$ so $g'=2x$. $f(g)=\sin(g)$ so $f'=\cos(g)$. So the derivative of $\sin(x^2)=2x\cos(x^2)$

Slope function of $\sin(x^2)$

See right. The solid curve function $\sin(x^2)$ is a sine, so it is bounded by ± 1 . As x gets bigger, x^2 changes more quickly, so the sine 'bunches up'. x^2 is an even function, and so is $\sin(x^2)$

The dotted slope function is $2x\cos(x^2)$. The \cos also bunches up as x gets bigger. But it is multiplied by $2x$, so the envelope of the slope also increases.



Example Two

What is the derivative of $\cos(2x)$?

The *quick way to do this* in your head is to differentiate the inner part (which is $2x$, so you get 2) and the outer part (which is \cos , so you get $-\sin$, leaving it as a function of $2x$) and multiply, so you get

$$-2\sin(2x)$$

Derivative of $f(ax)$

We can generalise this to find the derivative of $f(ax)$, where a is a constant. The derivative of ax is just a , so the **derivative of $f(ax)$ is $af'(ax)$** .

For example, the derivative of $\sin(3x)$ is $3\cos(3x)$

Example Three

What is the derivative of $\sin^2(x)$?

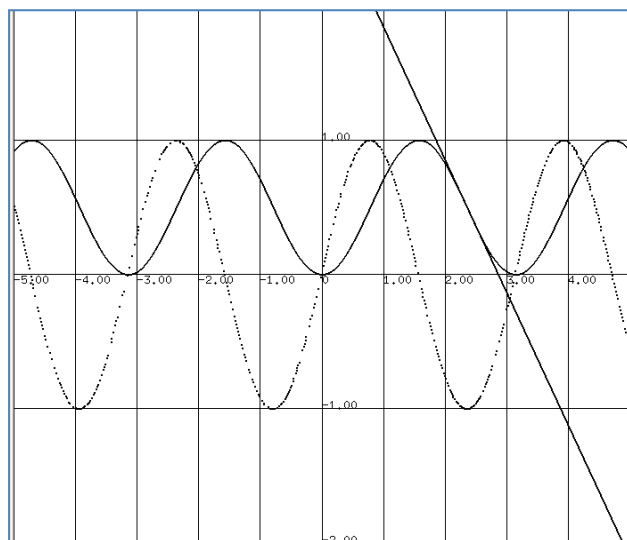
This is the same as the first example, but f and g are swapped. We first find $\sin(x)$, then square it. So

$$g = \sin(x) \text{ so } g' = \cos(x)$$

$$f = g^2 \text{ so } f' = 2g = 2\sin(x)$$

and the derivative of $\sin^2(x)$ is $2\sin(x)\cos(x)$

We could do this by a different route using some trig identities (we wouldn't because it's longer, but it's a check):



$$\sin^2 x =$$

$$\frac{1}{2}(1 - \cos(2x)) =$$

$$\frac{1}{2} - \frac{\cos 2x}{2}$$

and the derivative of this, using the chain rule on $\cos 2x$ as in example 2, is

$$0 - \frac{-2\sin(2x)}{2}$$

$$= +\sin(2x) = 2\sin(x)\cos(x)$$

Exercise

Find the derivatives of

$$4\cos(3x)$$

$$8\sin(2x) + 2\cos(2x+1)$$

$$5x + \cos(2x)$$

$$\sin(\cos(x))$$