Exponentials and logarithms

Exponentials

We have looked at functions like x^2. What happens if we swap over the x and the two, so that we have 2^x? Here are some values:

x	-2	-1	0	1	2	3	4
2^x	1/4	1/2	1	2	4	8	16

This gives some idea of what the graph would look like. The actual shape is shown in Figure One, together with a tangent, and the derivative as dots.

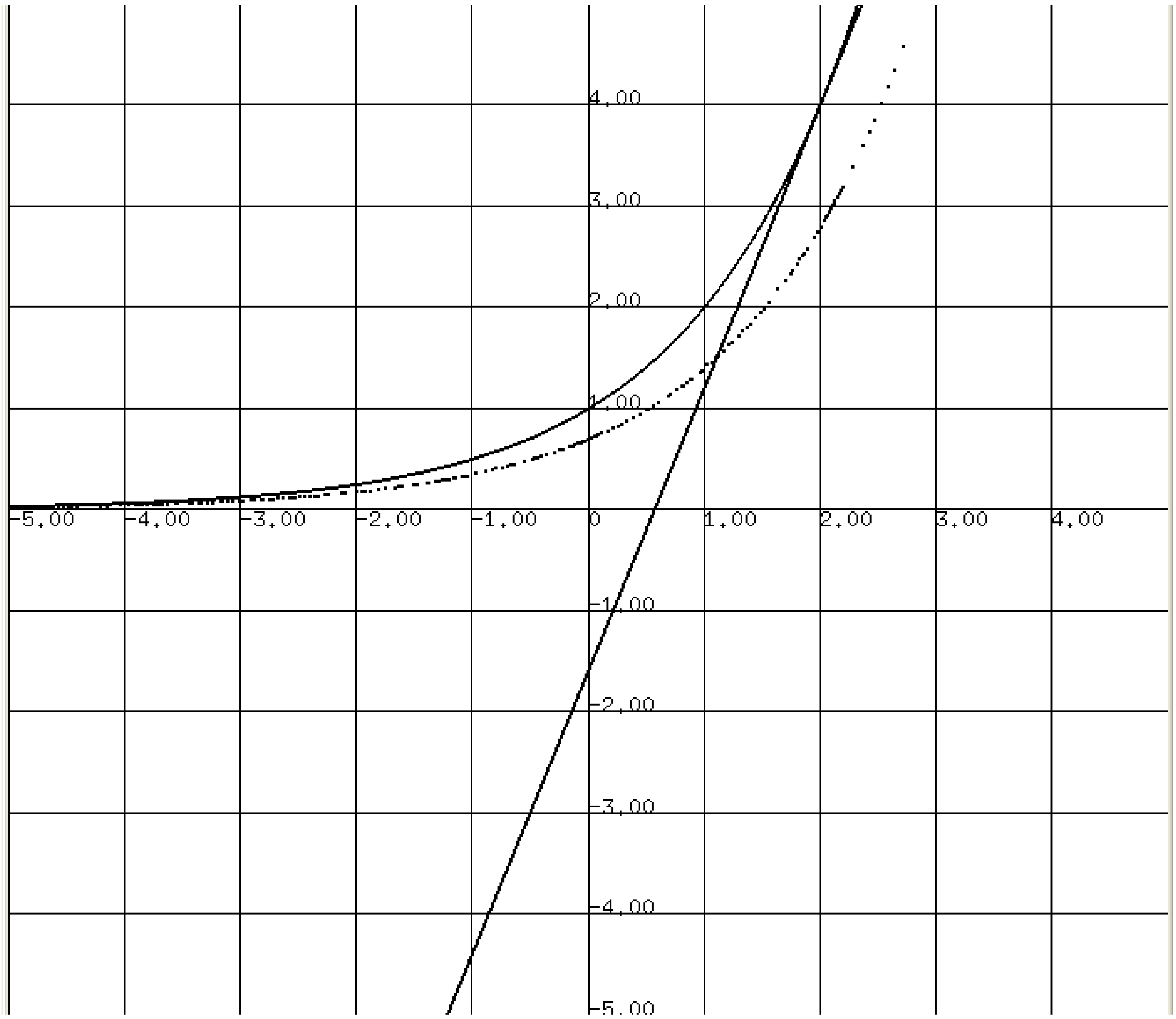
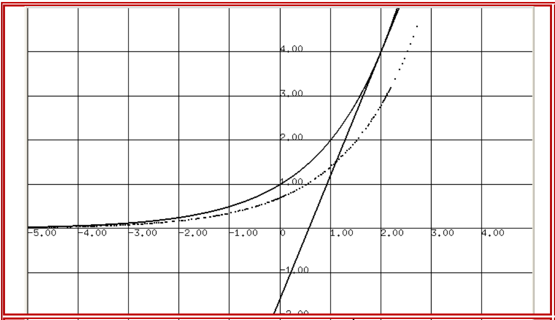


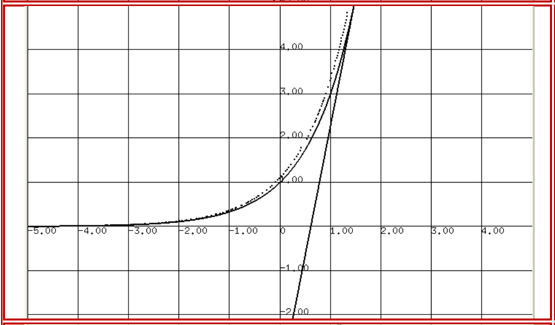
Figure 1

As well as 2^x, we might have 3^x, 4^x, 1.5^x, 2.1^x and so on. Functions of the form k^x where k is a constant are called exponential functions.

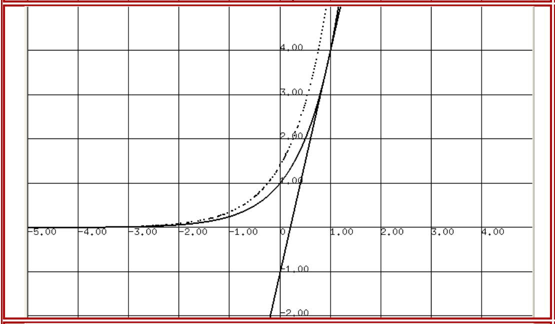
On the next page is displayed the graphs of 2^x, 3^x, 4^x and 5^x, together with their derivatives:



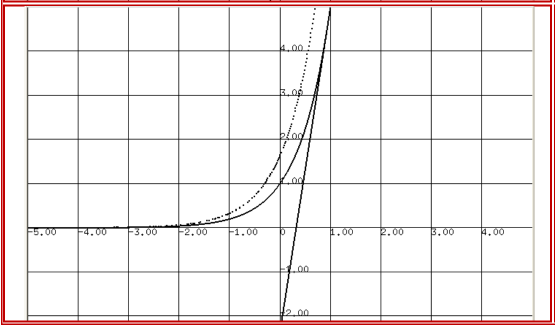
k=2



k=3



k=4



k=5

For all these functions, their derivatives are roughly the same as the function itself. But for k=2, the derivative is slightly less than the function, and for 3, 4 and 5 the derivative is more than the function.

This suggests we could find a value for k, slightly less than 3, where the derivative was equal to the function.

Exercise One

Use graphical software to try to find this number k so that the derivative of k^x is the same as k^x.

The number e

This number is called ‘e’, sometimes known as Napier’s number or Euler’s number. It’s value is about 2.718 – e is actually irrational. The number e occurs throughout mathematics, like π. There are several definitions of e, but we will use this one – e is that number such that the derivative of e^x is equal to e^x.

Examples Two

Combining this with ideas we’ve seen before, we can differentiate more functions:

Function	Derivative	Why
$2e^x$	$2e^x$	deriv. of af(x) = af’
e^{3x}	$3e^{3x}$	deriv. of f(ax) = af’
$e^x + e^{-x}$	$e^x - e^{-x}$	deriv. of f+g = f’ + g’, and chain rule
$2x + e^{3x}$	$2+3e^{3x}$	deriv. of f+g = f’ + g’, and deriv. of f(ax) = af’, and chain rule
$e^x\sin x$	$e^x\sin x+e^x\cos x = e^x(\sin x+\cos x)$	product rule

Derivative of a^x

Suppose we want the derivative of a^x, where 'a' is a constant other than e? We can work this out using the identity

$$a^x = \quad e^{x \ln a}$$

This is a function of a function. The derivative of x ln(a) is ln(a). So the derivative of a^x is $\ln(a)e^{x\ln(a)}$

Derivatives of Inverse functions

Recall that if f(x)=y, then x=f^{-1}(y), where f^{-1} is the inverse of the function f.

Now by definition

$$\frac{dy}{dx} = f'$$

so

$$\frac{dx}{dy} = \frac{1}{f'}$$

and this is the derivative of f^{-1}. So

The derivative of the inverse = 1/derivative of the function

Let's try an example: f(x)=2x+3, so f'=2

The inverse f^{-1}=(x-3)/2. So the derivative of the inverse is 1/2

Remember that some functions do not have inverses.

Derivatives of logarithmic functions

The logarithm is the inverse function of e^x.

In other words if f(x)=y=log x

then $x=e^y$

Now (thinking of x as a function of y)

$$\frac{dx}{dy} = e^y = x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

so the derivative of log x is 1/x

Examples Three

1) What is the derivative of log(3x)? This is a function (log) of a function (3x), so we use the chain rule, and the derivative is $3\frac{1}{3x} = \frac{1}{x}$

2)What is the derivative of log(x^2)? This is 2log(x), so the derivative is 2/x