

## Derivatives of products

Suppose we have two functions – how can we find the derivative of the product?

For example, what is the derivative of  $x\sin(x)$ ? We can differentiate both  $x$  and  $\sin(x)$  – but what if they are multiplied?

### The Product Rule

If the functions are  $u(x)$  and  $v(x)$ , then the derivative of the product  $uv$  is

$$vu' + uv'$$

Learn this.

### Example One

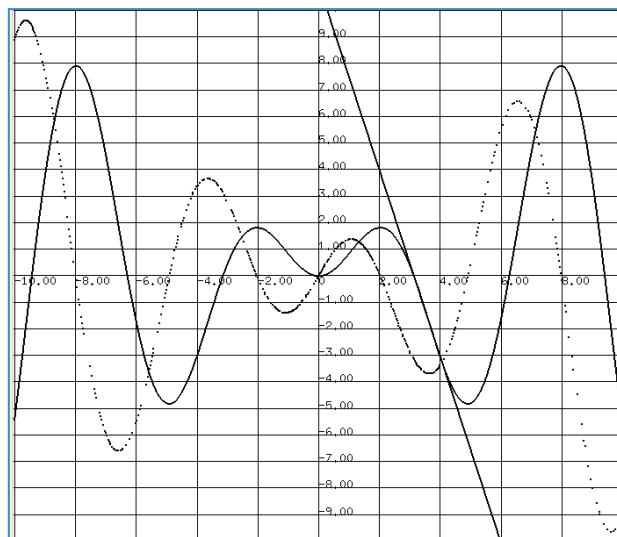
To find the derivative of  $x\sin(x)$ , so that  $u(x)=x$  and  $v(x)=\sin(x)$ , then

$$vu' = \sin(x).1 = \sin(x)$$

$$uv' = x.\cos(x)$$

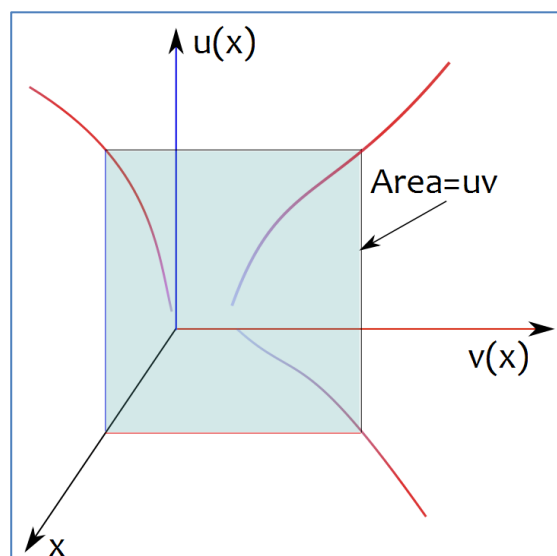
so the derivative of  $x\sin(x)$  is  $\sin(x)+x\cos(x)$

This is shown on the right – the solid curve is  $x \sin x$ , the dots are the slope function and the straight line is a tangent.



### Visualization

We can visualise the two functions, and their product, as on the right in 3D perspective. The  $x$  axis is running 'on the floor' and forward. Function  $u$  of  $x$  is plotted vertically up the side wall. Function  $v$  of  $x$  is plotted horizontally across the floor. The product  $uv$  at any value of  $x$  is the area of the rectangle as shown.



Remember the generic tangents. We assume  $u$  and  $v$  are locally straight. Then  $du = u' dx$  and  $dv = v' dx$ . We can split the change in area  $d(uv)$  into three parts: a top slice  $vdu$ , a right hand vertical slice  $udv$ , and the top right corner  $dudv$ . The change in area is the sum:

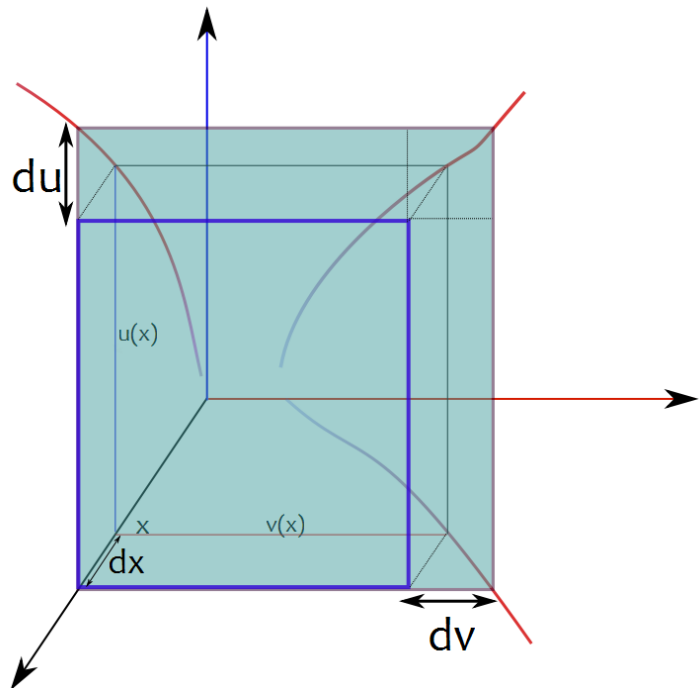
$$d(uv) = vdu + udv + dudv$$

$$= vu' dx + uv' dx + dudv$$

The generic tangents are best when  $h$  is small – and so is  $dx$ ,  $du$  and  $dv$ . The product  $dudv$  will be extremely small, and we can neglect it. So

$$d(uv)/dx = vu' + uv'$$

and this is our measure for the derivative of  $uv$ .



### Example Two

What is the derivative of  $x^2 \cos(x)$ ?

This is a product, of  $x^2$  and  $\cos$ . So the derivative is

$$x^2 (-\sin(x)) + 2x \cos(x) =$$

$$2x \cos(x) - x^2 \sin(x)$$

### Example Three

What is the derivative of  $5x^2 + 2x^3 \sin x$  ?

This is a sum of two terms, which we can differentiate separately. The second term is a product of two functions. So the derivative is

$$10x + 6x^2 \sin x + 2x^3 \cos x$$

$$= 2x(5 + 3x \sin x + 5x^2 \cos x)$$