Reciprocals and Quotients

Reciprocals

How can we differentiate the reciprocal of a function, such as 1/sin(x)?

A reciprocal, or 1 over something, is a function. So this is a function of a function – our example is the reciprocal of sin x. So we should be able to use the chain rule to differentiate it.

To find the derivative of a reciprocal, we can use the rule for differentiating x^n, where n=-1. So this is just $-1/x^2$ .But we are not differentiating x. We are differentiating a function v(x). So using the chain rule, the derivate of 1/v(x) is

$$\frac{-1}{v^2}.v' = \frac{-v'}{v^2}$$

Example One

What is the derivative of 1/sin(x)? This is usually called cosec or csc.

Our v(x)=sin(x).

So v' = cos(x)

and the derivate of 1/sin(x) = $\frac{-\cos x}{sin^2 x} = \frac{-1}{\sin x}.\frac{\cos x}{\sin x} = -\csc x \ \cot x$

Example Two

What is the derivative of 1/cos(x), usually called sec.

v(x)=cos(x), so v' = -sin(x)

and the derivate of 1/cos(x) = $\frac{--\sin x}{cos^2 x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x} = \sec x \ \tan x$

Derivatives of Quotients

In other words, how do we differentiate one function over another, like $x^2/\sin(x)$?

In general this is u(x)/v(x), which is the product of u(x) and 1/v(x). We can differentiate products, and reciprocals, so here we go – the derivative of u/v is

$$u\frac{-v'}{v^2} + u'v =$$

$$\frac{-uv' + u'v}{v^2}$$

$$= \frac{u'v - v'u}{v^2}$$

Example Three

What is the derivative of $x^2/\sin(x)$?

u = x^2 so u' is 2x

v is sin(x) so v' is cos(x)

Putting these in the formula, we get

$$\frac{2x \sin x - x^2 \cos x}{\sin^2 x} =$$

$$x\left(\frac{2 - x\ cot(x)}{\sin x}\right)$$

We could have seen $x^2/\sin(x)$ as a product of x^2 and csc(x), and since the derivative of csc is –csc cot, this is

$$-x^2 \csc x \cot x + 2x csc(x) =$$

$$x \csc x\ (-x \cot x + 2) =$$

$$x\left(\frac{2 - x \cot x}{\sin x}\right)$$

Example Four

This gives us a way of differentiating tan(x), since this is a quotient sin(x)/cos(x)

u=sin(x) and u'=cos(x)

v=cos(x) and v'=-sin(x)

So the derivative of tan(x) is

$$\frac{\cos x\ \cos x + \sin x \sin x}{\cos^2 x}$$

$$\frac{1}{\cos^2 x} = \sec^2 x$$

Example Five

Similarly for cot x =cos x /sin x

u=cos x and u'= -sin x

v=sin x and v'= cos x

So the derivative of cot x is

$$\frac{-\sin x \ \sin x - \cos x \cos x}{\sin^2 x}$$

$$\frac{-1}{\sin^2 x} = -\csc^2 x$$