Reciprocals and Quotients

Reciprocals

How can we differentiate the reciprocal of a function, such as $1/\sin(x)$?

A reciprocal, or 1 over something, is a function. So this is a function of a function – our example is the reciprocal of $\sin x$. So we should be able to use the chain rule to differentiate it.

To find the derivative of a reciprocal, we can use the rule for differentiating $x^n$, where $n=-1$. So this is just $-1/x^2$. But we are not differentiating $x$. We are differentiating a function $v(x)$. So using the chain rule, the derivative of $1/v(x)$ is

$$\frac{-1}{v^2} \cdot v' = \frac{-v'}{v^2}$$

Example One

What is the derivative of $1/\sin(x)$? This is usually called cosec or csc.

Our $v(x) = \sin(x)$.

So $v' = \cos(x)$

and the derivative of $1/\sin(x) = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$

Example Two

What is the derivative of $1/\cos(x)$, usually called sec.

$v(x) = \cos(x)$, so $v' = -\sin(x)$

and the derivative of $1/\cos(x) = \frac{-\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$

Derivatives of Quotients

In other words, how do we differentiate one function over another, like $x^2/\sin(x)$?

In general this is $u(x)/v(x)$, which is the product of $u(x)$ and $1/v(x)$. We can differentiate products, and reciprocals, so here we go – the derivative of $u/v$ is

$$u \frac{-v'}{v^2} + u'v = $$
Example Three
What is the derivative of \( \frac{x^2}{\sin(x)} \)?

\( u = x^2 \) so \( u' \) is 2x

\( v = \sin(x) \) so \( v' \) is \( \cos(x) \)

Putting these in the formula, we get

\[
\frac{2x \sin(x) - x^2 \cos(x)}{\sin^2(x)} = \]

\[
x \left( \frac{2 - x \cot(x)}{\sin(x)} \right)
\]

We could have seen \( \frac{x^2}{\sin(x)} \) as a product of \( x^2 \) and \( \csc(x) \), and since the derivative of \( \csc \) is \( -\csc \cot \), this is

\[-x^2 \csc x \cot x + 2x \csc(x) =
\]

\[
x \csc x (-x \cot x + 2) =
\]

\[
x \left( \frac{2 - x \cot(x)}{\sin(x)} \right)
\]

Example Four

This gives us a way of differentiating \( \tan(x) \), since this is a quotient \( \sin(x)/\cos(x) \)

\( u = \sin(x) \) and \( u' = \cos(x) \)

\( v = \cos(x) \) and \( v' = -\sin(x) \)

So the derivative of \( \tan(x) \) is

\[
\frac{\cos x \cos x + \sin x \sin x}{\cos^2(x)}
\]

\[
\frac{1}{\cos^2(x)} = \sec^2(x)
\]

Example Five

Similarly for \( \cot(x) = \cos(x) / \sin(x) \)

\( u = \cos(x) \) and \( u' = -\sin(x) \)
\( v = \sin x \) and \( v' = \cos x \)

So the derivative of \( \cot x \) is

\[
\frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x
\]