Integration by Parts

What is it?

This is a technique of integration which is sometimes useful when we want to integrate a product of two functions, such as

$$I = \int x \sin x \, dx$$

Here the product is the 'first part' (the function x) multiplied by the 'second part' (the function sin(x)).

This can be worked out as follows (the reason is given next):

first times integral of the second, minus the integral of the differential coefficient of the first times the integral of the second.

Learn this by heart.

For our example, this is

x	-cos x	-	$\int$	1	-cos x
first	integral of the second	minus	the integral of	differential coefficent of the first	integral of the second

that is,

$$I = \int x \sin x \;\, dx =$$

$$x(-\cos x) - \int 1.(-\cos x) \, dx$$

$$= -x \, cos \, x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

where C is a constant of integration.

We can check this by differentiating it and hope to get back to the original x sin x:

$\frac{d}{dx}(-x \cos x + \sin x + C) =$	(use product rule on –x cos x .. vdu/dx+udv/dx)
$x \, sin \, x - cos \, x + cos \, x + \; 0 =$	

$x \sin x$	

Why does it work?

This is based on the rule for differentiating a product. Suppose we want to integrate the product of two functions of x, called u and g:

$$I = \int u(x)g(x)dx$$

The product rule applied to u() and g() is:

$$\frac{d}{dx}ug = u\frac{dg}{dx} + g\frac{du}{dx}$$

Integrate both sides of this with respect to x:

$\int \frac{d}{dx}ug = \int u\frac{dg}{dx} + \int g\frac{du}{dx}$	
$ug = \int u\frac{dg}{dx} + \int g\frac{du}{dx}$	by the Fundamental Theorem of Calculus : ∫d/dx f(x) dx = f(x)
$\int u\frac{dg}{dx} = ug - \int \frac{du}{dx}g$	simple re-arrangement

Suppose we define a third function v(x):

$$v(x) = \frac{dg}{dx}$$

$$so\ g = \int v(x)dx$$

so our integral can be written:

$$\int uv = u\int v - \int \frac{du}{dx}\int v$$

We have called u the first part and v the second part. This is the same as

first times integral of the second, minus the integral of the differential coefficient of the first times the integral of the second.

Learn this by heart

How to use it – second example

To find

$$I = \int \ln(x) x^2 dx$$

This is

ln(x)	$\frac{x^3}{3}$	-	$\int$	$\frac{1}{x}$	$\frac{x^3}{3}$
first	integral of the second	minus	integral of	differential coefficient of the first	integral of the second

so

$$I = \ln(x)\frac{x^3}{3} - \int \frac{1}{x}\frac{x^3}{3}\, dx$$

$$= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C$$

$$= \frac{x^3}{3}\left(\ln(x) - \frac{1}{3}\right) + C$$

Who goes first?

In the last example, we could have exchanged the first and second terms, or in other words written the integral as

$$I = \int x^2 \ln(x) dx$$

The 'integral of the second', ln x, is x ln(x)-x. This is derived in the next section. So the integral is

$$x^2(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x}) - \int 2\mathrm{x}(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x})\mathrm{dx}$$

This looks more complicated than the original. Always we have a choice of which is first and which is second. Usually one choice works and the other does not.

The factor of unity trick

Any expression can be written as a product, namely of 1 and the expression. For example, ln(x) could be written as 1.ln(x), or as ln(x).1. So

$$I = \int \ln(x)\, dx = \int \ln(x) . 1\, dx$$

$$= \ln(x)\, x - \int \frac{1}{x} x dx \qquad (by\ parts)$$

$$= \ln(x)\, x - \int 1 dx$$

$$= x\ln(x) - x + C$$

which is a result we used in the last section.

Going round in circles

Sometimes using integration by parts produces an intermediate result, and when used a second time, produces an expression which includes the original integral. It looks like we are back where we started, but this can be re-arranged to give the answer we want.

For example,

$$I = \int e^x \cos(x)\, dx$$

$$= e^x \sin(x) - \int e^x \sin(x)\, dx \qquad (intermediate\ result - do\ it\ again)$$

$$= e^x \sin(x) - \left(\mathrm{e}^{\mathrm{x}} \left(-\cos(x)\right) - \int e^x \left(-\cos(x)\right) dx\right)$$

$$= -e^x \sin(x) + \mathrm{e}^{\mathrm{x}}\ \cos(x) - \int e^x \cos(x)\, dx$$

This has a term which is our original integral I, so

$$I = -e^x \sin(x) + \mathrm{e}^{\mathrm{x}}\ \cos(x) - I$$

$$so\ 2I = -e^x \sin(x) + \mathrm{e}^{\mathrm{x}}\ \cos(x)$$

$$and\ I = \frac{e^x}{2}\left(\cos(\mathrm{x}) - \sin(\mathrm{x})\right)$$

Round in circles again

We tried to integrate $x^2\ln(x)$ by parts this way round, obtained the result:

$$I = x^2(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x}) - \int 2\mathrm{x}(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x})\mathrm{dx}$$

and gave up. In fact this is

$$I = x^2(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x}) - 2\int \mathrm{x}^2 \ln(\mathrm{x})\,\mathrm{dx} + \int 2\mathrm{x}^2\,\mathrm{dx}$$

$$I = x^2(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x}) - 2\mathrm{I} + \frac{2\mathrm{x}^3}{3}$$

$$3I = x^2(\mathrm{x}\ln(\mathrm{x}) - \mathrm{x}) + \frac{2\mathrm{x}^3}{3}$$

$$3I = x^3\ln(\mathrm{x}) - \frac{\mathrm{x}^3}{3}$$

$$I = \frac{x^3}{3}\left(\ln(x) - \frac{1}{3}\right)$$

So in fact we can integrate this as $x^2 \ln(x)$ or $\ln(x) x^2$, but the latter is easier.

Reduction formulae

A reduction formula takes the integration of an expression which includes a power of a trig function, and expresses it as something having a lower power. For example

$$I = \int sin^n(x)dx$$

$$= \int sin^{n-1}(x). \sin(x)dx$$

$$= -\sin^{n-1}(x)\cos(x) + \int (n-1)sin^{n-2}(x)\cos(x)\cos(x)\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + \int (n-1)sin^{n-2}(x)(1-\sin^2(x))dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int sin^{n-2}(x)dx - (n-1)\int sin^n(x)dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int sin^{n-2}(x)dx - (n-1)I$$

$$so\ nI = -\sin^{n-1}(x)\cos(x) + (n-1)\int sin^{n-2}(x)dx$$

$$and \int sin^n(x)dx = \frac{-\sin^{n-1}(x)\cos(x)}{\mathrm{n}} + \frac{(n-1)}{n}\int sin^{n-2}(x)dx$$

A Visual Approach to Integration by Parts

We want to evaluate an integral which is written as a product, of the ‘first part’ u(x) and the ‘second

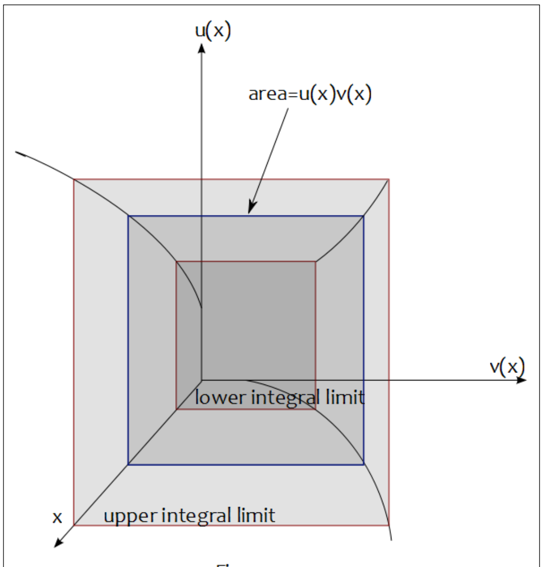

Figure 1

part’ v(x):

$I = \int u(x)v(x)dx$

We can see these two functions of x together (Figure 1), if we draw u(x) vertically on the 'left-hand wall', and v(x) horizontally on the 'floor', with the independent variable x axis along the edge of the

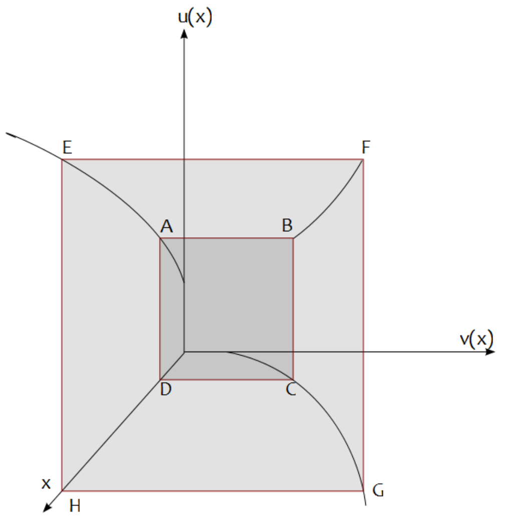

Figure 2

wall and the floor.

The value of u(x)v(x) at any value of x is the area of a rectangle. The integral I is therefore the volume

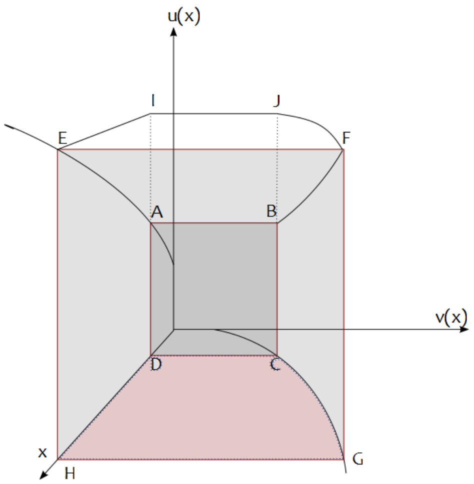

Figure 3

ABCDEFGH in Figure 2.

The usual integration by parts formula is

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int u' \int v(x)dx\, dx$$

Can we find graphical forms of the two terms?

The v(x) integral is the area under the curve – or since our graph is sideways, the area to the left of the curve (the pink area in Figure 3). Multiplying it by u gives the volume IJCDEFGH.

What about the second term? We can see this more effectively from a different point of view – looking horizontally towards the wall. In figure 4 we see a finite change in x, δx, producing a change in u, δu=δxtanθ. On the floor we have an area ∫vdx, and the vertical height δu sweeps out a volume element δu∫vdx, above the curve EA. In the limit as δx→0, tanθ→du/dx.

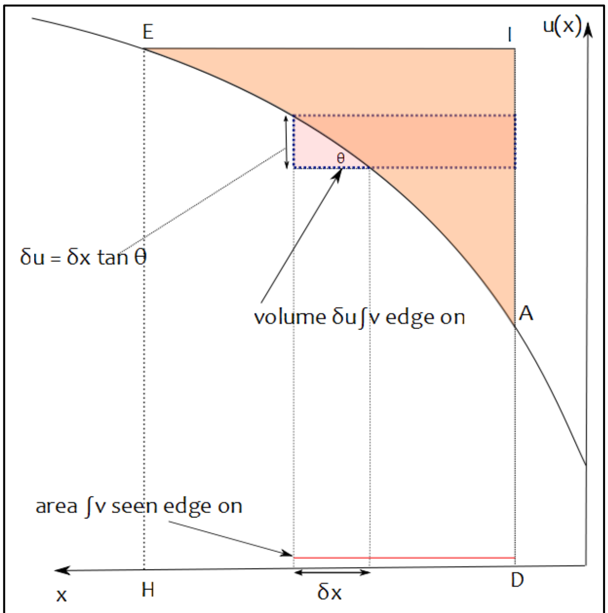

Figure 4

The limit of the sum of the finite volume elements as δx →0 is the orange coloured area, which is

$$\int u' \int v(x)dx\, dx$$

So in

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int u' \int v(x)dx\, dx$$

the first term on the right is the volume IJCDEFGH in figure 3, and the second is the volume above this, IJBAFE.