Area under the curve

The integral of a function is the area under its curve, between two limits. If the function is f, and the independent variable is x, (so f is a function of x), and the limits are 'a' and 'b', this is written like this:

$$I = \int_{x=a}^{x=b} f(x)dx$$

If it is obvious what the independent variable, the 'x=a' is often written as just 'a'.

This is the shaded area in figure 1.

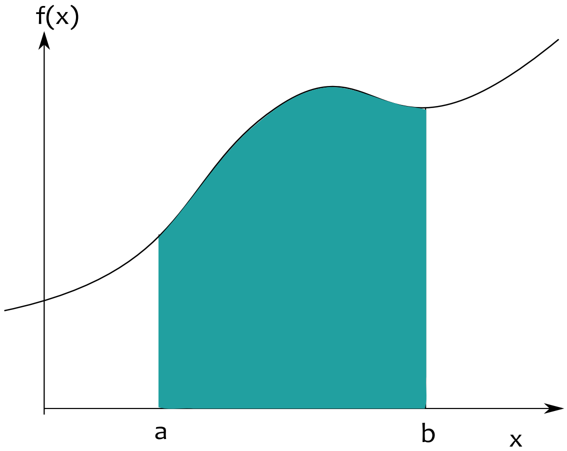


Figure 1

Example 1

Suppose our function f(x) = 3x, and we need to integrate this from x=0 to x=2. In other words we need to find

$$I = \int_0^2 3x\, dx$$

This is the area in Figure 2:

This is simply a triangle, and the area = ½ base times height, so it is 6. This is the value of this integral.

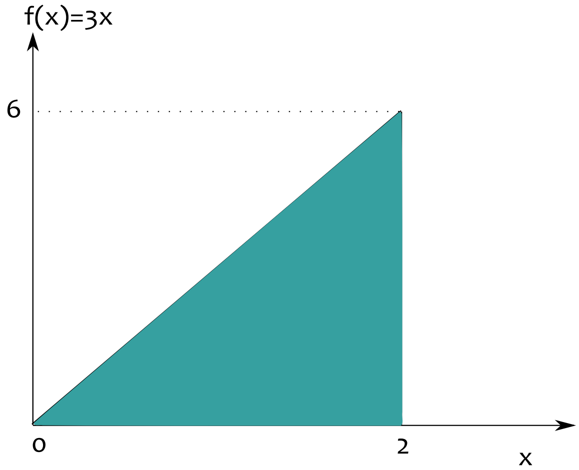


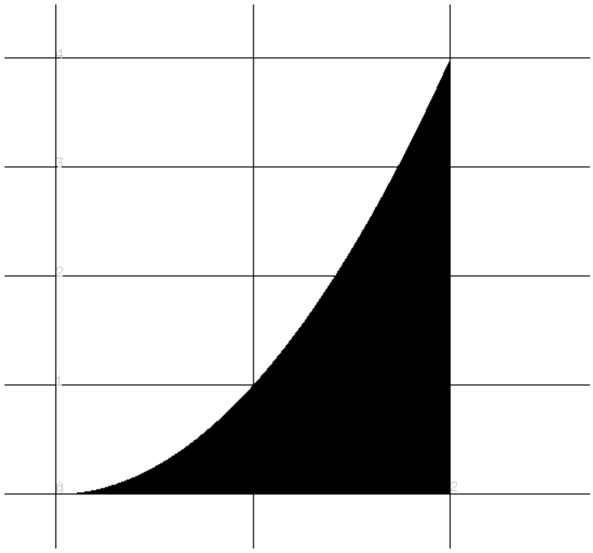
Figure 2

Example 2

Suppose we want to find

$$\int_0^2 x^2\,dx$$

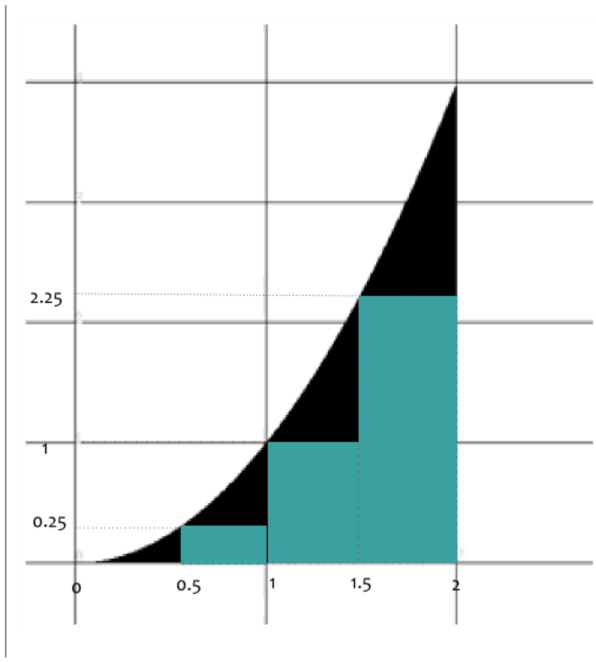
This is shown on the right.



This is a problem – it's a curve, not a simple triangle, and so there is no obvious way of working out the area.

Using vertical strips

We could get a rough value if we divided the x interval into four, and drew vertical strips up to the curve, as on the right. Each strip has width 0.5. The columns have been drawn so that their height is equal to the curve at the lower end of each strip. So the first strip, with x going from 0 to 0.5, has height zero.



The areas of each strip are:

0.5 X 0 = 0

0.5 X 0.25 = 0.125

0.5 X 1 = 0.5

0.5 X 2.25 = 1.125

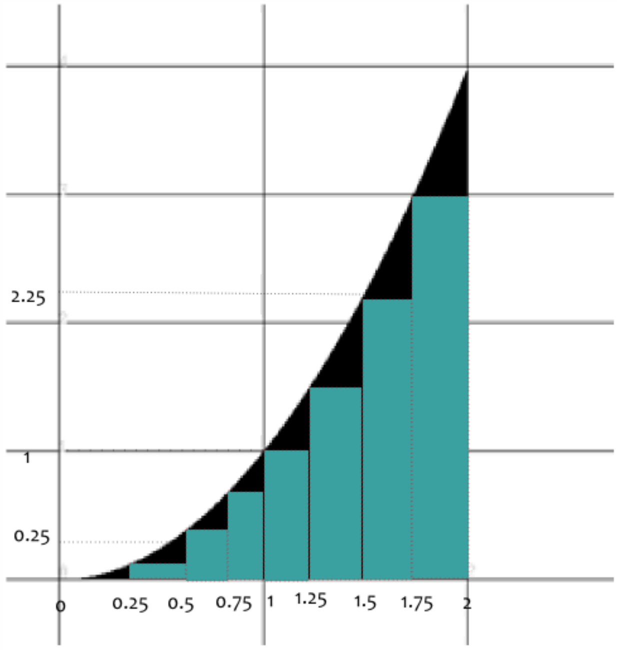
so the total area of the strips = 0.125+0.5+1.125 = 1.750

This under-estimates the actual area under the curve, which is 2.667 – but it is a first approximation

More and more strips

If we use more strips, we'll clearly get a more accurate answer, since those 'undershoots' will be smaller. For example, using eight strips is shown below.

If you add up the areas of these strips, you get 2.1875 – closer to the actual value of 2.667.



If we use more strips, we get a more accurate value. If we use infinitely many strips, will we get a perfect answer? Each strip would have zero width, and so zero area. But we would have infinitely many of them – so would the total be finite?

We are not clear what we mean by infinity. But we might agree that if a number n becomes very large, 1/n becomes very small. And if n becomes infinitely large, 1/n becomes zero.

So the plan is

1. Suppose we have n strips

2. Work out an algebraic expression for the areas of those strips

3. Work out the sum of their areas.

4. Let n become indefinitely large, and see what value the sum becomes.

For sums, we use the sigma notation. Sigma is a Greek letter, $\sum$ as a capital letter, and corresponds to the English letter ‘s’, for sum. For example

$$\sum_{m=1}^{3} m^2$$

means the sum of m^2, as m goes from 1 to 3. In other words, 1+4+9 = 14

We will want to find

$$\lim_{n\to\infty} \sum_{m=1}^{n} area\ of\ the\ mth\ strip$$

which is the limit as n tends to infinity of the sum of the n strips.

Infinitely many strips

We will work this out for

$$\int_0^a x^2\,dx$$

which is the area under the curve of x^2 from 0 to a constant 'a'.

We have n strips.

So the width of each strip is a/n

m numbers the strips, going from 0 to n-1. So the left-hand edge of the strips are at x=0, x=a/n, x=2a/n, x=3a/n .. and the last one at (n-1)a/n.

The mth. strip starts at x=ma/n.

The height of the mth. strip is x^2, or

$$\frac{m^2a^2}{n^2}$$

The area of the mth. strip is width X height, or

$$\frac{m^2a^2}{n^2}\frac{a}{n} = \frac{m^2a^3}{n^3}$$

The total area is

$$\sum_{m=0}^{n-1}\frac{m^2a^3}{n^3}$$

As m changes (0, 1, 2..n-1) a and n do not change. each term in the sum has a^3/n^3 as a common factor, and this can be factored out:

$$\frac{a^3}{n^3}\sum_{m=0}^{n-1}m^2$$

The sum is equal to

$$\frac{(n-1)n(2(n-1)+1)}{6}$$

This result is proved below. So our area is

$$\frac{a^3}{n^3}\frac{(n-1)n(2(n-1)+1)}{6} =$$

$$\frac{a^3}{6n^3}(n-1)n(2(n-1)+1) =$$

$$\frac{a^3}{6n^3}(n^2-n)(2n-2+1) =$$

$$\frac{a^3}{6n^3}(n^2 - n)(2n - 1) =$$

$$\frac{a^3}{6n^3}(2n^3 - 2n^2 - n^2 + 1) =$$

$$\frac{a^3}{3} - \frac{a^3}{2n} + \frac{a^3}{6n^3}$$

This is the sum of the areas of n strips. What happens to this as n tends to infinity?

$$\lim_{n \to \infty}\left(\frac{a^3}{3} - \frac{a^3}{2n} + \frac{a^3}{6n^3}\right)$$

The second term has n in the denominator, so this becomes zero. So does the third term. The first term is independent of n. So the value of this limit is just $a^3/3$, which is our integral:

$$\int_0^a x^2\,dx = \frac{a^3}{3}$$

Integrals in general

If we had to integrate all functions like this, algebraically, life would be tough. There are two alternatives:

1. Calculating it numerically, for a very large number of strips. This is hard work by hand, but easy on a computer. This is called *numerical integration*. The good news is that any sensible function can be integrated this way. The bad news is that it is not exact.
2. Finding the integral as the anti-derivative. In other words, finding a function which when you differentiate it, you get the function you are trying to integrate. This is described in a separate section. For example, if you differentiate $x^3/3$ you get x^2 – which is a lot easier than all that algebra. The good news is that the answer is exact. The bad news is that it is often impossible to find such a function . The second good news is that this gives us a new way to actually define some new functions.

Nevertheless, we should still think of the meaning of an integral as

$$\lim_{n \to \infty} \sum_{m=1}^{n} \textit{area of the mth strip}$$

in other words as a sum of an infinite number of elements. This is why $\int$ is used as the integral sign – it is an eighteenth century letter ‘s’, for sum.

Proof of the sum of n^2

We want to prove that

$$\sum_{m=0}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$$

We prove this by induction. In other words we prove it is true for n=1. Then we prove that if it is true for n, it is true for n+1. Hence it is true for all n.

If n=1, the formula gives $\frac{1\ X\ 2\ X\ 3}{6} = 1$, which is correct. So it is true for n=1

If it is true for n, is it true for n+1?

$$\sum_{m=1}^{n+1} m^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{(n^2+n)(2n+1)}{6} + n^2 + 2n + 1$$

$$= \frac{2n^3+3n^2+n}{6} + n^2 + 2n + 1$$

$$= \frac{2n^3+3n^2+n+6n^2+12n+6}{6}$$

$$= \frac{2n^3+9n^2+13n+6}{6}$$

$$= \frac{2n^3+6n^2+4n+3n^2+9n+6}{6}$$

$$= \frac{(n^2+3n+2)(2n+3)}{6}$$

$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

which is the corresponding formula for n+1.

So it is true for all n.