The Fundamental Theorem of Calculus

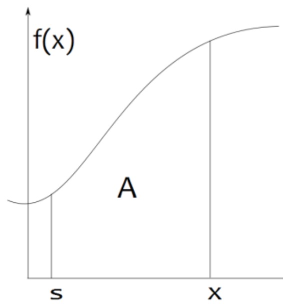
So far we have seen two unrelated ideas:

1. $\int_a^b f(x)dx$ is defined to be the area under the graph of f(x) between a and b
2. $\int f(x)dx = g(x)\ implies \frac{d}{dx}g(x) = f(x)$ - the integral is the anti-derivative

It is not obvious that the anti-derivative has anything to do with area, and so it is not clear why both these ideas use the integral sign.

But..

Consider the area under the graph of f(x), marked A in the diagram on the right. This is the area between an arbitrary starting point 's', and another point 'x', which we allow to vary. As x changes A will also change, so that A is a function of x:

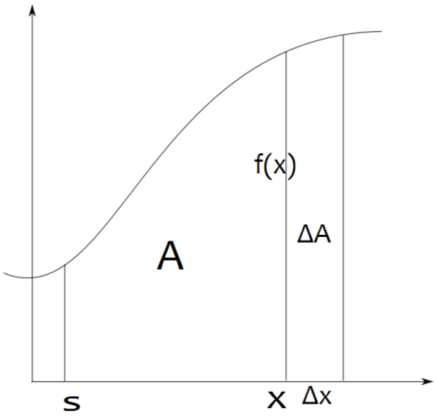


This

$$\int_s^x f(x)dx = A(x)$$

Now suppose x increases by an amount Δx.

The resulting change in area, ΔA, is approximately a rectangle, with width Δx and height f(x)



$$\Delta A \approx \Delta x f(x)$$

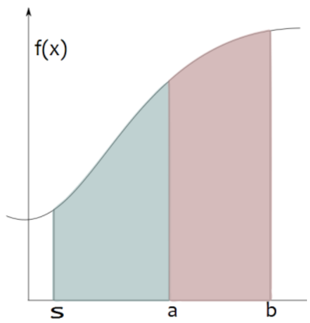
$$\frac{\Delta A}{\Delta x} \approx f(x)$$

In the limit,

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx} = f(x)$$

But comparing this with (2) above, this means our area function A(x) is the same as the anti-derivative g(x).

Suppose we want to find the area between two limits, from a to b (see right). Clearly the area from a to b will be the area from s to b, less the area from s to a. Our area function g(x) is the area up to x. In other words



$$\int_a^b f(x)dx = g(b) - g(a)$$

The form g(b)-g(a) is often written:

$$\left[g(x)\right]_a^b$$

Example

Find the area under the curve of x^2 between x=1 and x=2.

The anti-derivative of x^2 is $x^3/3$, so the area is

$$\int_1^2 x^2\,dx =$$

$$[\frac{x^3}{3}]_1^2 =$$

$$\frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

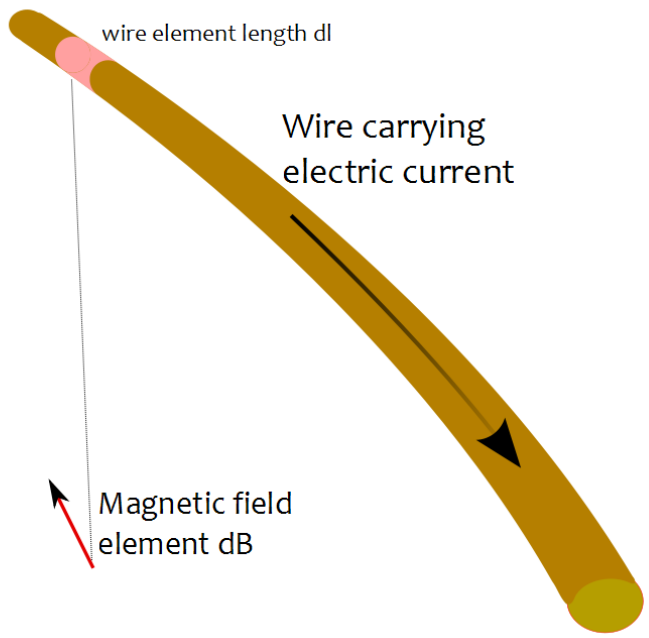
The anti-derivative and the constant of integration

In fact the anti-derivative also has a constant of integration. But when we find the difference of the anti-derivative at the two limits, the constants will disappear, and when finding areas this way we can forget about them.

Who cares about areas?

The Fundamental Theorem of Calculus and its associated way of finding areas is enormously important in science and engineering, as well as mathematics. If it were only about finding areas, this would seem strange. But the key aspect of a definite integral is that it is a sum – of area strips maybe, but also the sum of other things. Three examples are outlined.

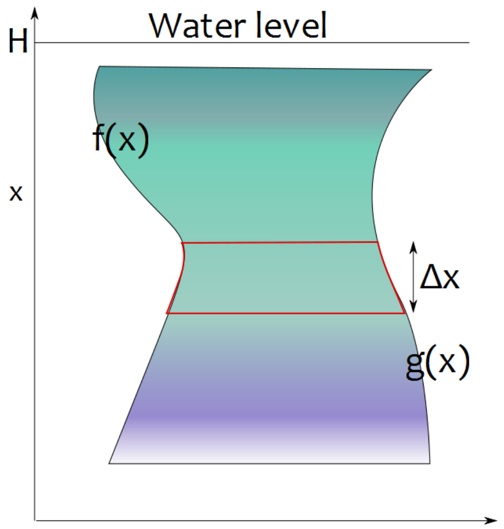
Magnetic field due to an electric current



The Biot-Savart Law lets us calculate the magnetic field dB caused by an electric current in a short (in fact, infinitesimal) length of wire dl, as in the diagram. But to find the overall magnetic field at that point, we need to add up the effects of all the elements down the length of the wire. Finding the sum of those infinitesimal elements means integration.

Hydrostatic force on a plate in water

Suppose we want to find the force on a plate suspended vertically in water (maybe as a first step in calculating the force on a submarine hull). The left and right edges of the plate are given by functions f and g, as on the right.



The pressure at depth h in a fluid of density ρ is ρgh, where g is the acceleration due to gravity.

The area of an element of the plate, as shown, is about Δx(g(x)-f(x). So the force = pressure X area=ρg(H-x) (g(x)-f(x)) Δx

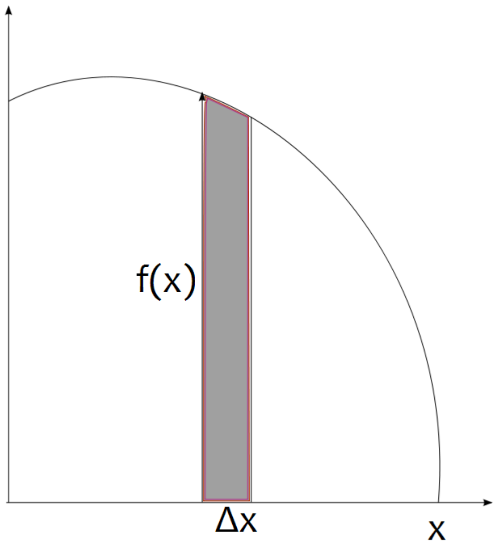
As Δx tends to zero, this becomes exact – we find the total force on the plate by finding the sum of these force elements:

$$\rho g \int (H - x)(g(x) - f(x)dx$$

The limits of the integration are the top and bottom of the plate.

Finding the mass of a lamina

Suppose we need to calculate the mass of a lamina (flat sheet) made out of metal with uniform density ρ, when the shape of the lamina is described by some function f(x).



If the sheet has unit depth, then the mass per unit area is ρ. The area of a vertical strip is approximately Δx f(x), and so the mass of this strip is about ρ Δx f(x). In the limit this is exact, and the mass of the sheet is ρ ∫f(x)dx, between limits to match the width of the sheet.

Exercise

1. Sketch the graph of f(x)=x+4

2. What is $\int_1^2 (x+4)dx$?

3. Try to reason (do not calculate, yet) whether the following integrals will be positive or negative:

a) $\int_0^1 (x+4)dx$

b) $\int_{1}^{0}(x+4)dx$

c) $\int_{-1}^{0}(x+4)dx$

d) $\int_{0}^{-1}(x+4)dx$

e) $\int_{-5}^{-4}(x+4)dx$

f) $\int_{-4}^{-5}(x+4)dx$

4) Calculate the integrals in question 4 to check your reasoning